MATH2050C Assignment 5

Deadline: Feb 11, 2025.

Hand in: 3.3. no. 5, 10; Suppl. Problems no. 1, 2.

Section 3.3 no. 3, 5, 7, 10.

Supplementary Problems

1. Show that the sequence $\{b_n\}, b_n = \sum_{k=1}^{n} \frac{1}{k^a}$ is convergent iff and only if $a > 1$. Hint: Study b_{2^n} as in Example 3.3.3b in textbook.

2. Show that (a) $x_n = (1 + 1/n)^n$ is strictly increasing and $y_n = (1 + 1/n)^{n+1}$ is strictly decreasing. Hint: Try the Bernoulli inequality.

3. Show the limit of $(1 - 1/n)^n$ as $n \to \infty$ is equal to $1/e$. Hint: Use Problem 3 in Ex 4.

4. Prove that e is irrational. Hint: Use the inequality $0 < e-(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}) < \frac{1}{k\times k!}$.

The exponential

The monotone convergence theorem gives a criterion for the existence of limits for sequences. It is very useful especially when the limit is not easy to guess.

Theorem 5.1 Monotone Convergence Theorem An increasing sequence is convergent iff it is bounded above. A decreasing sequence is convergent iff it is bounded below.

The proof is to show when an increasing sequence is bounded above, it converges to its supremum (whose existence is ensured by the Order Completeness Property of $\mathbb{R}$), see our textbook for details. You can also fine some good applications of this theorem from the book.

The exponential e and π are the most fundamental constants in mathematics. e is defined as the limit of an increasing sequence whose root lies on natural growth or compound interest.

Theorem 5.2 For $a > 0$, the sequence $x_n = (1 + a/n)^n$ is strictly increasing and bounded from above. Consequently,

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$$

exists.

Proof By binomial theorem,

$$\begin{aligned} x_n &= 1 + n\frac{a}{n} + \frac{n(n-1)}{2!}\frac{a^2}{n^2} + \cdots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{a^k}{k!} + \cdots + \frac{a^n}{n^n} \\ &= 1 + a + \frac{1}{2!}\left(1-\frac{1}{n}\right)a^2 + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)a^k + \\ &\quad \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)a^n . \end{aligned}$$

x_{n+1} is obtained by replacing the n's in the formula above by $n+1$. By a term by term comparison, we see that $x_n < x_{n+1}$, that is, $\{x_n\}$ is strictly increasing. Next, from this formula we also have

$$x_n < 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!} .$$

Observe that for some large n_0, $a^n/n! \le (1/2)^n$ for all $n \ge n_0$, hence

$$1+a+\frac{a^2}{2!}+\cdots\frac{a^{n_0-1}}{(n_0-1)!}+\frac{a^{n_0}}{n_0!}\cdots \le 1+a+\frac{a^2}{2!}+\cdots\frac{a^{n_0-1}}{(n_0-1)!}+\frac{1}{2^{n_0}}+\cdots = 1+a+\frac{a^2}{2!}+\cdots\frac{a^{n_0-1}}{(n_0-1)!}+\frac{2}{2^{n_0}}.$$

It shows that $1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!}$ is bounded by some M for all n. By Monotone Convergence Theorem the limit of x_n exists and is equal to its supremum.

For $a \ge 0$, we define a function $E(a)$ by setting

$$E(a) = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n .$$

We also write $e = E(1)$ and call it the exponential.

Theorem 5.3 For each k,

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right) \le \frac{1}{k \times k!} .$$

Proof From the proof above, for $k < n$,

$$\begin{aligned}
0 &< \left(1+\frac{a}{n}\right)^n - \left(1+a+\frac{1}{2!}a^2+\frac{1}{3!}a^3\cdots+\frac{1}{k!}a^k\right) \\
&< \frac{a^k}{k!}\left(\frac{a}{k+1}+\frac{a^2}{(k+2)(k+1)}+\frac{a^3}{(k+3)(k+2)(k+1)}+\cdots\right) \\
&< \frac{a^k}{k!}\left(\frac{a}{k+1}+\frac{a^2}{(k+1)^2}+\frac{a^3}{(k+1)^3}+\cdots\right) \\
&= \frac{a^k}{k!}\frac{a}{k+1}\frac{1}{1-a/(k+1)} \\
&= \frac{a^{k+1}}{k!}\frac{1}{k+1-a}\ .
\end{aligned}$$

By letting $n \to \infty$, we obtain the desired conclusion where strict inequality in both side still holds. The result follows by taking $a = 1$.